SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 1

In the following problems, a "fake n-manifold" is a topological space M which is locally Euclidean. That is, for every point $p \in M$, there exists a neighborhood $U \subset M$ of p, an open set $V \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$.

Problem 1. Let M be a connected topological n-manifold, and $C \subset M$ be a closed proper subset. Let M^f be the set

$$M^f = \left\{ (x, a) : x \in M, \text{ and } \begin{array}{ll} a = 0, & x \notin C \\ a = \pm 1, & x \in C \end{array} \right\}$$

Define a topology on M^f as being generated by two types of open sets from derived from the topology on M: If $U \subset M$ is open, let

- $U^{\#} = \{(x, 0) : x \in U \setminus C\} \cup \{(x, 1) : x \in C \cap U\}$, and
- $U^{\flat} = \{(x, 0) : x \in U \setminus C\} \cup \{(x, -1) : x \in C \cap U\}$.

Show that, with the topology generated by sets of the form $U^{\sharp}$ and $U^{\flat}$, M^f is a fake n-manifold, but not a manifold. What happens if $C = M$?

Remark 1. In the previous problem, when $M = \mathbb{R}$ and $C = \{0\}$, this is often called the "line with two origins."

Problem 2. Give an example of a Hausdorff fake n-manifold which is not a manifold (and justify why it is not a manfiold). [*Hint*: A disjoint union of locally Euclidean spaces is still locally Euclidean]

Remark 2. The example you come up with in the previous problem is probably not connected. For a connected example, look up a pathology called the *long line*.

Problem 3. Show that $S^n = \left\{ x \in \mathbb{R}^{n+1} : ||x|| = 1 \right\}$ has a canonical smooth n-manifold structure by explicitly finding a smooth atlas and showing the atlas is smooth.

Problem 4. Show that if M is a smooth m-manifold and N is a smooth n-manifold, then $M \times N$ has a canonical smooth $(m + n)$-manifold structure.

Problem 5. Show that $\mathbb{R}^2 \setminus \{0\}$, $A = \left\{ x \in \mathbb{R}^2 : 1 < ||x|| < 2 \right\}$ and $S^1 \times (0, 1)$ are all diffeomorphic with their standard smooth structures. [*Hint*: Find explicit diffeomorphisms between them. Show that the maps you find are bijective and differentiable, with invertible derivative.]

Problem 6. Let X denote the boundary of the unit square in $\mathbb{R}^2$. Prove or find a counterexample:

(1) X is a topological 1-manifold.
(2) There exists a smooth structure on X.
(3) There exists a smooth structure on X such that the inclusion of X into $\mathbb{R}^2$ is C^∞.
(4) There exists a smooth structure on X such that the inclusion of X into $\mathbb{R}^2$ is an immersion.

Linear algebra and vector calculus review

Problem 7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and assume that there exists a $v \in \mathbb{R}^n$ such that v is an eigenvector of $dF(x)$ with real eigenvalue for every $x \in \mathbb{R}^n$. Show that the lines $L(x) = \{x + tv : t \in \mathbb{R}\}$ are *equivariant*: $F(L(x)) = L(F(x))$.

Problem 8 (Contraction mapping principle, differentiable version)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that the eigenvalues of $DF(x)$ all have modulus at most some $\lambda < 1$ for every $x \in \mathbb{R}^n$. Show that F has a unique fixed point x_0, and for every $x \in \mathbb{R}^n$, $F^k(x) \to x_0$ as $k \to \infty$.

Problem 9. Let V and W be (real) finite-dimensional vector spaces and $\mathrm{Hom}(V, W)$ be the set of linear transformations from V to W.

(1) Show that $\mathrm{Hom}(V, W)$ is a real vector space.
(2) With fixed bases for V and W, find an isomorphism between $\mathrm{Hom}(V, W)$ and $M(m, n)$, the set of $m \times n$ matrices, where $m = \dim(V)$ and $n = \dim(W)$.
(3) If $V_0 \subset V$ is a subspace of V, let $\mathrm{Ann}(V_0) \subset \mathrm{Hom}(V, W)$ be the *annihilator* of V_0. That is, the set of $\varphi \in \mathrm{Hom}(V, W)$ such that $\varphi(v) = 0$ for all $v \in V_0$. Show that $\mathrm{Ann}(V_0)$ is a vector subspace of $\mathrm{Hom}(V, W)$, then find and prove a formula for $\dim(\mathrm{Ann}(V_0))$ in terms of $\dim(V)$, $\dim(W)$ and $\dim(V_0)$. [*Hint*: It might be useful to think about it as matrices using the previous part]
(4) * Find a canonical isomorphism between $V^* \otimes W$ and $\mathrm{Hom}(V, W)$, and prove it is an isomorphism. Construct a projection $\pi : V^* \otimes W \to V_0^* \otimes W$ such that $\mathrm{Ann}(V_0) = \ker \pi$, and prove that it is a projection, and that the kernel is as described. Deduce the formula for $\dim(\mathrm{Ann}(V_0))$ using π, as well.